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Is It Possible to Derive Newtonian Equations of Motion with Memory?

R.R. Nigmatullin · D. Baleanu

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Abstract In this paper for a given example we proved that the Riemann-Liouville fractional integral term appears naturally and relates the external force with acceleration within the fractional Newtonian equation. The consideration of some self-similar process that leads to the fractional integral as well as some possible generalizations of the proposed model was discussed.

Keywords Fractional calculus · Newtonian equation · Riemann-Liouville fractional integral

1 Introduction

The fractional calculus is as old as the classical ones and it deals with derivative and integral of any order [1–4, 6, 7, 9–12, 14, 15]. The properties of the fractional derivatives and integrals are not as general as the classical ones; therefore we expect at the first sight that this kind of calculus has less potential applications than the one we know. However, the applications of the fractional calculus reveal some rich type of problems, which can be solved by using the fractional calculus. The fractional calculus describes better the behavior of complex systems involving memory effect. During the last decades the researchers tried to build fractional models in several areas of science and engineering. The validation of the fractional calculus will come when a series of phenomena will be described only within this

R.R. Nigmatullin
Theoretical Physics Department, Kazan State University, Kremlevskaya str., 18, Kazan, 420008, Tatarstan, Russian Federation
e-mail: nigmat@knet.ru

D. Baleanu (✉)
Department of Mathematics and Computer Science, Cankaya University, 06530 Ankara, Turkey
e-mail: dumitru@cankaya.edu.tr

D. Baleanu
Institute of Space Sciences, P.O. BOX, MG-23, 76900, Magurele-Bucharest, Romania

type of calculus. This question is as deep as the one related to the physical meaning of the fractional integrals and derivatives. In our opinion the answer to this question is one of the key points of the fractional calculus. One way is to look to some behaviors of the complex phenomena, to apply the fractional techniques and to conclude that within this technique the results are better and look more general to the classical ones. Another way is more solid and more accurate to deal with physical non-local phenomena and it can be summarized as follows: we have to start from the basic laws of physics and general concepts of physics, to fractionalize them properly and to prove that there is a physical meaning of the obtained results. In other words, we have to find a class of physical phenomena which requires naturally the fractional calculus derivatives and integrals.

Quite recently, [13] V.V. Uchaikin published his book entitled as "The Methods of the Fractional Derivatives", where he collected the numerous examples of application of the mathematical apparatus of the fractional calculus starting from nanophysics, from one side and moving to the fractional structure of the Universe, from another one. In consideration of many physical systems two basic points have to be taken into account: the importance and influence of the memory (nonlocal) effects in time and space and the finding of the general physical meaning of non-integer operators of differentiation and integration. These two basic ideas will help to make a decisive step for penetration of the mathematics of the fractional calculus into 'a body' of natural sciences. In spite of the fact that many researches are trying to apply the mathematical methods of the fractional calculus in their own investigations (bibliography of the papers sited in [13] exceeds more than 1000 papers, reviews and books) the absence of the clear physical/geometrical meaning serves a specific obstacle on the way of applications. One of us (RRN) [5, 8] suggested a rather simple explanation of the fractional integral with real and complex power-law exponents that will help on the first stage to more comprehensive interpretations of these non-integer operators. As for the experimental confirmation of existence the fractional kinetics in Mother-Nature we should remark the set of papers of one of us (RRN) related to development of the theory of dielectric relaxation based on the fractional kinetics. We remark here the recent publications that help to restore earlier ones [16–18].

The meaning of the fractional differentiation operation can be expressed by the following sentences:

The Riemann-Liouville fractional integral represents itself the continuous analog of the convolution operation of a smooth function with a fractal discrete set that modeling some self-similar process. If the discrete structure of the fractal set is not essential that the averaging procedure leads to the conventional definition of the RL integral with real power-law exponent. If the localization of neighboring stages of the process depends essentially on discrete structure then the imaginary part of the exponent of the RL integral reflects the influence of the discrete structure of the self-similar process considered in the continual limit.

In this paper we are going to show how to derive the Newtonian equations with memory taking as an example the relative movement of the layers of viscous liquid relatively each other. This simple idea should give a key idea how to derive the equations with memory if initially the memory effect seems to be absent.

2 Formulation a Problem and Physical Model

It is well known that initially the Newtonian equations were proposed as equations without memory

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}(\mathbf{r}, \mathbf{v}, t). \tag{1}$$

For one particle, we can formulate the following question: Is it possible to derive equation of the following type

$$m\frac{d\mathbf{v}}{dt} = \int_0^t K(t-\tau)\mathbf{F}(\mathbf{r},\mathbf{v},\tau)d\tau \tag{2}$$

when memory effects are taken into account. If this type of equations can be derived then it will open new possibilities of understanding of the classical mechanics. Here we want to describe a simple mechanical model that allows in realization of the desired transition from (1) to (2).

Description of the Mechanical Model. Let us consider N layers of viscous and heterogeneous liquid (layer polymer) and the movement of each kth laminar layer with density ρ_k $(k = 1, 2, \ldots, N)$ is described by the following Newton's equation

$$\begin{aligned}
&\rho_1\frac{d\mathbf{v}_1}{dt} = \gamma_{1,2}\mathbf{v}_2,\\
&\vdots\\
&\rho_k\frac{d\mathbf{v}_k}{dt} = \gamma_{k,k+1}\mathbf{v}_{k+1},\\
&\vdots\\
&\rho_{N-1}\frac{d\mathbf{v}_{N-1}}{dt} = \gamma_{N-1,N}\mathbf{v}_N,\\
&\rho_N\frac{d\mathbf{v}_N}{dt} = \mathbf{F}(|\mathbf{r}_{1,N}|,|\mathbf{v}_{1,N}|,t).
\end{aligned} \tag{3}$$

The physical meaning of this system of equations is the following. The movement of each layer is described presumably by movement of the neighboring layers located on the right-hand side. The first and the Nth layers occupy the final left and right-hand positions, correspondingly. So, the numeration of the layers is started from the left to the right. So, we take into account the influence of the layers, which are located closer to the acting external force (more precise to the density of the force). This force depends of the relative distance $|\mathbf{r}_{1,N}| = |\mathbf{r}_1 - \mathbf{r}_N|$ and relative velocity $|\mathbf{v}_{1,N}| = |\mathbf{v}_1 - \mathbf{v}_N|$ with respect to the relative movement and location of the first layer. Let us try to exclude the intermediate variables in order to reduce the influence of the external force to the effective movement of the first layer. Using the mathematical induction method it is easy to derive the equation of motion for the velocity $\mathbf{v}_1(t)$. It accepts the form

$$\begin{aligned}
\rho_1\frac{d\mathbf{v}_1}{dt} = {}& \frac{\gamma_{12}\gamma_{23}\cdots\gamma_{N-1,N}}{\rho_2\rho_3\cdots\rho_N}\frac{1}{(N-2)!}\int_0^t (t-\tau)^{N-2}F(|\mathbf{r}_{1,N}|,|\mathbf{v}_{1,N}|)d\tau\\
&+\sum_{k=2}^{N}\mathbf{v}_k(0)\frac{t^{k-2}}{(k-2)!}.
\end{aligned} \tag{4}$$

So, we obtain the Newton's equation with memory and the influence of all intermediate layers is reduced to the influence as some external factors. In [19] the classical derivatives were replaced by the fractional ones and some exact solutions were found. Based on the results obtained in the book [5] and later in paper [8] of one of the authors (RRN) is it easy

to transform the integer integral into the *fractional* one if one supposes that the temporal interaction between layers represents itself a self-similar process and distributed over the Cantor set and its possible generalizations. Because of importance and generality of this process it merits the separate consideration.

3 The Consideration of Some Self-Similar Process that Leads to the Fractional Integral

If we to take the Laplace-image of (4) then the further consideration is reduced to consideration of the convolution of the smothered external force with memory function that it will be presented in more general form

$$\rho\frac{d\mathbf{v}}{dt}=\int_0^t K(t-\tau)\mathbf{F}(\tau)d\tau \quad \overset{LT}{\Longleftrightarrow} \quad \rho\left[p\mathbf{v}(p)-\mathbf{v}(0)\right]=K(p)\mathbf{F}(p). \tag{5}$$

Here and below $\mathbf{v}(0)$ is defined as $\mathbf{v}(t=0)$. We consider the memory function $K(p)$ as the result of some self-similar process that takes place in the physical system considered. In this case the memory function $K(p)$ is written in the form of the product [15]

$$K(p)\equiv K^{(N)}(z)=\prod_{n=-(N-1)}^{N-1}\hat{g}(z\xi^n)=\prod_{n=0}^{N-1}\hat{g}(z\xi^n)\prod_{n=1}^{N-1}\hat{g}(z\xi^{-n}), \tag{6}$$

where $z=p\tau_0$ is the dimensionless parameter of the Laplace transform (τ_0 determines the characteristic time of the self-similar process at $n=0$), ξ—defines the value of the scaling factor. The product (6) defines the Cantor set with M bars or its possible generalization. In particular, if the self-similar process considered is distributed over the Cantor set with M bars that the function $g(z)$ accepts the form

$$g_M(z)=\frac{1}{M}\frac{1-\exp\left(-\frac{zM}{M-1}\right)}{1-\exp\left(-\frac{z}{M-1}\right)}. \tag{7}$$

In relationship (7) $z=pT(1-\xi)$, where T determines the temporal segment where the Cantor set is located, the value M determines the number of the Cantor's bars. The product (6) satisfies the following exact functional equation

$$K^{(N)}(\xi z)=\frac{g(z\xi^N)}{g(z\xi^{-N+1})}K^{(N)}(z). \tag{8}$$

For further investigation of the product (8) we suppose that the function $g(z)$ has the following decompositions, for $\mathrm{Re}(z)\ll 1$

$$\hat{g}(z)=1+c_1z+c_2z^2+\cdots, \tag{9a}$$

for $\mathrm{Re}(z)\gg 1$

$$\hat{g}(z)=\bar{g}+\frac{A_1}{z}+\frac{A_2}{z^2}+\cdots. \tag{9b}$$

Keeping the limiting values for both sides, from (8) we obtain the simplified functional equation

$$K(\xi z) = \frac{1}{\bar{g}} K(z). \tag{10}$$

The general solution of the functional equation (10) has the form [14, 15]

$$\lim_{N\to\infty} K^{(N)}(z) \equiv K_\nu(z) = \frac{\pi_\nu(\ln(z))}{z^\nu}, \tag{11}$$

where

$$\nu = \frac{\ln(\bar{g})}{\ln(\xi)} = \frac{\ln\left(\frac{1}{\bar{g}}\right)}{\ln\left(\frac{1}{\xi}\right)}, \tag{12}$$

and $\pi_\nu[\ln(z)]$ is a log-periodical function with the period $\ln\xi$

$$\pi_\nu(\ln(z) \pm \ln\xi) = \pi_\nu(\ln(z)). \tag{13}$$

So, we proved that the general solution for a self-similar process has the structure expressed by relationship (11). The further investigations realized in paper [8] showed that the log-periodical function is essential only for the cases when we have strong correlations (memory effects) between the neighboring location of the discrete stages of the self-similar process considered. When the self-similar process is presumably random (the neighboring stages of the process do not correlate with each other) then in the infinite series

$$\begin{aligned} K_\nu(z) &= \sum_{n=-\infty}^{\infty} C_n \exp\left[\left(\ln\left(\frac{1}{\bar{g}}\right) + 2n\pi i\right)\frac{\ln(z)}{\ln(\xi)}\right] \\ &= \sum_{n=-\infty}^{\infty} C_n \exp[(-\nu + i\Omega_n)\ln(z)], \end{aligned} \tag{14}$$

only the zero-Fourier component in (14) is conserved

$$K_\nu(z) \cong C_0 z^{-\nu}, \quad C_0 = \langle \pi_\nu(\ln(z)\rangle = \int_{-a}^{a} \pi_\nu(\ln z + x\ln\xi)dx. \tag{15}$$

Using the obvious relationship

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \overset{LT}{=} p^{-\alpha}, \tag{16}$$

we obtain from (4)

$$\begin{aligned} \rho_1 \frac{d\mathbf{v}_1}{dt} = {} & \frac{\gamma_{12}\gamma_{23}\cdots\gamma_{N-1,N}}{\rho_2\rho_3\cdots\rho_N} \frac{1}{\Gamma(N-2+\nu)} \int_0^t (t-\tau)^{N-2+\nu} F(|\mathbf{r}_{1,N}|, |\mathbf{v}_{1,N}|)d\tau \\ & + \sum_{k=2}^{N} \mathbf{v}_k(0) \frac{t^{k-2}}{(k-2)!}. \end{aligned} \tag{17}$$

Here the power-law exponent ν is supposed to be located in the interval $[0, 1]$, $\Gamma(x)$ is the ordinary gamma-function. Here we want to remark that similar equation was obtained in [20].

The authors used the same ideas of the averaging of a smooth function but the self-similar set represented itself the generalization of the conventional Cantor set.

So, based on the results of this section (see more details in [5, 8]) it is possible to derive the Newtonian equations with memory. The memory function for some self-similar process is expressed in the form of the desired Riemann-Liouville integral.

4 Possible Generalizations

It is possible to generalize the system of equations (3) taking into account the interaction between velocities of *all* layers that are located on the right-hand side with respect to the initial layer

$$\begin{aligned}
&\rho_1 \frac{d\mathbf{v}_1}{dt} = \gamma_{1,2}\mathbf{v}_2 + \gamma_{1,3}\mathbf{v}_3 + \cdots + \gamma_{1,N}\mathbf{v}_N,\\
&\vdots\\
&\rho_k \frac{d\mathbf{v}_k}{dt} = \gamma_{k,k+1}\mathbf{v}_{k+1} + \cdots + \gamma_{k,N}\mathbf{v}_N,\\
&\vdots\\
&\rho_N \frac{d\mathbf{v}_N}{dt} = \mathbf{F}(|\mathbf{r}_{1,N}|, |\mathbf{v}_{1,N}|, t).
\end{aligned} \tag{18}$$

It is easy to prove that this system also leads to the structure (5), where the memory function has the following structure

$$K(t) = \sum_{k=1}^{N-2} r_k \frac{t^k}{k!}, \tag{19}$$

where the coefficients r_k form a certain ratio from the coefficients of interaction $\gamma\{k\}$ and densities $\rho\{k\}$. The concrete form of these coefficients is not so important for this analysis. For us it is important also the fact that this model leads again to the structure of the Newtonian's equations with memory. As before, the intermediate layers form a specific 'field' of interaction between the applied force and the velocity of the first layer.

5 Conclusions

The fractional calculus has several deep open problems. Despite of the fact that several attempted were made to clarify the geometrical and the physical meaning of the fractional integral up to now there is no general answer to this problem. Therefore, all models based on the replacement of the classical derivatives with the fractional ones have in some cases, a fundamental question from the beginning: what is the physical meaning of the model?

Fractional calculus is an emerging field and the corresponding models are non-local ones. Therefore, it is natural to ask if we can find easily a model, which is non-local and exists in our daily life. The first think is to ask: what is meaning of the fractional Newtonian mechanics and what is the non-local counterpart of the second law of mechanics. Going back to the roots of the fractional mechanics we can ask ourselves to find a process which is governed by a non-local Newton's second law. Actually, the Riemann-Liouville integral (the fractional derivative is obtained easily by using the fractional derivative) is obtained after a

repetitive process which leads naturally to a convolution of an external force with memory function generated by external field that relates the acceleration with the force. We want to stress here again that memory function is formed in the result of elimination of intermediate layers. These layers form a specific field of interaction that relates the external force with movement of the first layer.

Having these results in mind we investigated in the first part of the letter the Newton's equation applied to consideration of movement of N layers of viscous and heterogeneous liquid and the corresponding movement of each kth layer ($k = 1, 2 \ldots, N$) with a specified density. By using the mathematical induction method we have proved that the second law is given by (4), which is transformed to the Newton's equation with memory. However, the kernel contains only integer power. The next step was to find the extension to the fractional ones. The main problem was which way we have to use, namely a natural mathematical transformation of the natural number N into a real one or to prove that it exists a physical phenomena which is behind the corresponding fractional generalization.

In the second part of the letter we have proved that the result presented by (4) can be generalized to the fractional one under the assumption that the temporal interaction between the layers represents itself some general self-similar process and it is distributed over the Cantor set and its possible generalizations.

The results obtained in this paper open the door for discussion the origin of the fractional Newton's equation as a natural and real extension of the classical counterpart. Also, the results obtained here give a strong sign that the fractional calculus is a powerful natural mathematical tool to describe complex phenomena with memory effect.

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